Exact Model Reduction for Discrete-Time Conditional Quantum Dynamics

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What are the minimal resources needed to reproduce a target quantum process?

Finding the minimal resources allows us to reduce the model's description.

Why is quantum model reduction interesting?

- Efficient **quantum simulation** (on classical and quantum computers);
- Efficient implementations of:
  - **controllers**,
  - **error suppression schemes**,
  - **quantum filters**;
- Easier models to study;
- Proving optimality of quantum algorithms;
- Probing "quantumness" of processes;
- Efficient generation of quantum trajectories (Monte Carlo methods).

Dynamics of interest: Conditional dynamics

- CPTP evolution followed by generalized measurement;
- Imperfect measurement;
- Dynamics conditioned on the measurement outcome.

In general we assume to have a set of CP $\{\mathcal{M}_k\}_{k\in\Omega}$, s.t $\sum_{k\in\Omega}\mathcal{M}_k^\dagger(\mathbf{1}) = \mathbf{1}$ with Ω the set of possible outcomes.

If the state of the system is ρ, the outcome $k\in\Omega$ is observed with probability

$$\mathbb{P}_\rho[\mathcal{M}=k] = \mathrm{tr}[\mathcal{M}_k(\rho)].$$

The state of the system, conditioned on the measurement outcome, then is

$$\rho|_{\mathcal{M}=k} = \frac{\mathcal{M}_k(\rho)}{\mathrm{tr}[\mathcal{M}_k(\rho)]}.$$

Linear dynamics

The conditioning rule $\rho|_{\mathcal{M}=k} = \frac{\mathcal{M}_k(\rho)}{\mathrm{tr}[\mathcal{M}_k(\rho)]}$ is non-linear.

But we can consider **un-normalized states** $\tilde{\rho}$ so that the dynamics becomes

$$\tilde{\rho}(t+1) = \mathcal{M}_{k_{t+1}}[\tilde{\rho}(t)]\ .$$

The probability of a sequence of outcomes then is:

$$\mathbb{P}[\mathcal{M}_{0:t} = k_{0:t}] = \mathrm{tr}[\mathcal{M}_{k_{0:t}}(\rho_0)] = \mathrm{tr}[\mathcal{M}_{k_t} \circ \cdots \circ \mathcal{M}_{k_1} \circ \mathcal{M}_{k_0}(\rho_0)]$$

To compute the normalized state or expectation values one can simply re-normalize:

$$\rho(t) = \frac{\tilde{\rho}(t)}{\mathbb{P}[\mathcal{M}_{0:t} = k_{0:t}]} \qquad \langle O(t) \rangle = \mathrm{tr}[O\rho(t)] = \frac{\mathrm{tr}[O\tilde{\rho}(t)]}{\mathbb{P}[\mathcal{M}_{0:t} = k_{0:t}]}$$

Quantities of interest:

In many practical settings, one is not interested in the entire state ρ.

We thus assume to only be interest in reproducing the expectation value of a **set of observables of interest** $\{O_j\}$ that include the identity, $\mathbf{1} \in \{O_j\}$.

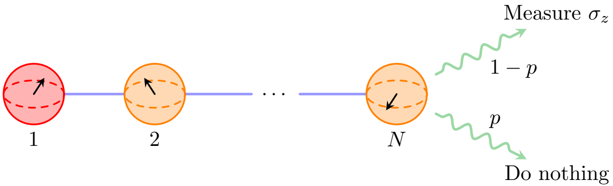
We can compactly represent these as a linear output map $\mathcal{C} : \mathfrak{B}(\mathcal{H}) \to \mathscr{Y}$:
$\mathcal{C}[\rho] = \sum_j E_j \mathrm{tr}[O_j \rho]$.

In the following we focus in the **conditional dynamics (CD)** $(\{\mathcal{M}_k\}, \mathcal{C})$:

$$\begin{cases} \tilde{\rho}(t+1) & = \mathcal{M}_{k_{t+1}}[\tilde{\rho}(t)] \\ Y(t) & = \mathcal{C}[\tilde{\rho}(t)] \end{cases}$$

Similar to Hidden Markov Models.

Monitored spin chain
- A toy model -



$$\mathcal{E}(\cdot) = e^{-iH} \cdot e^{iH} \qquad \text{where} \qquad H = \delta \sum_{j=1}^{N-1} \sigma_x^{(j)} \sigma_x^{(j+1)}$$

We assume to perform a projective on the last spin with probability $p \in (0,1)$.

$$\mathcal{K}_{-1}(\cdot) = p\mathbf{1} \cdot \mathbf{1}, \qquad \mathcal{K}_0(\cdot) = (1-p)\Pi_0 \cdot \Pi_0, \qquad \mathcal{K}_1(\cdot) = (1-p)\Pi_1 \cdot \Pi_1$$

where Π_0, Π_1 are the eigenprojectors of $\sigma_z^{(N)}$.

$$\mathcal{M}_k = \mathcal{K}_k \circ \mathcal{E}$$

We are actually interested in reproducing the evolution of the first spin, i.e.

$$\rho_1(t) = \mathrm{tr}_{\bar{1}}[\rho(t)] \qquad \mathcal{C}(\cdot) = \mathrm{tr}_{\bar{1}}(\cdot)$$

The problem: Quantum model reduction

Given a CD $(\{\mathcal{M}_k\}, \mathcal{C})$ find:

- **another CD** $(\{\check{\mathcal{M}}_k\}, \check{\mathcal{C}})$;
- a linear map $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{r\times r}$ so that $\check{\rho}_0 = \Phi[\rho_0]$

such that

$$\mathcal{C}[\mathcal{M}_{k_{0:t}}(\rho_0)] = \check{\mathcal{C}}[\check{\mathcal{M}}_{k_{0:t}}(\check{\rho}_0)]$$

for all $t \geq 0$ all $\rho_0 \in \mathfrak{D}(\mathcal{H})$, and for all sequences of outcomes $k_{0:t}$.

Observable space

Let us define

$$\mathscr{O} = \mathrm{span}\{O_j, \mathcal{M}^{\dagger}_{k_{0:t}}(O_j), \quad \forall j, \quad \forall k_{0:t}, \quad \forall t\}$$

the (Krylov) operator space that contains the observables of interest evolved in Heisenberg picture, for all possible trajectories.

One can see it as the orthogonal to the “usual” non-observable subspace $\mathscr{N}$.

FACT: $\mathscr{O}$ is the smallest $\mathcal{M}^{\dagger}_{k}$-invariant operator space that contains $\{O_j\}$.

$\mathscr{O}$ contains all the degrees of freedom we need to reproduce $\mathcal{C}[\mathcal{M}_{k_{0:t}}(\rho_0)]$.

Problem:
Restricting the model onto $\mathscr{O}$ does not ensure Complete Positivity

How do we ensure that physical (probability) constraints are satisfied?

The solution: $*$-algebras

We define a $*$-algebra $\mathscr{A}$ as an operator space closed under matrix multiplication and adjoint action.

$$X, Y \in \mathscr{A} \quad \Rightarrow \quad X + Y \in \mathscr{A} \qquad X^\dagger, Y^\dagger \in \mathscr{A} \qquad \text{and} \qquad XY \in \mathscr{A}$$

It is the fundamental mathematical structure that supports a
quantum probability space.

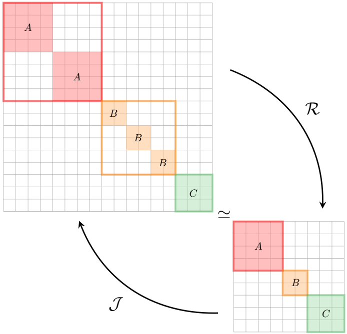
Conditional expectations

A *conditional expectation* $\mathbb{E}_{\mathscr{A},\rho}[\cdot]$ is a **CP unital proj. onto a $*$-algebra $\mathscr{A}$** (and such that $\mathrm{tr}[\rho\,\mathbb{E}_{\mathscr{A},\rho}[X]] = \mathrm{tr}[\rho X]$ for all $X \in \mathcal{B}(\mathcal{H})$).

When $\mathbf{1} \in \mathscr{A}$, $\mathbb{E}_{\mathscr{A},\mathbf{1}/n}[\cdot]$ exists and is an orthogonal proj. hence CPTP.

$\mathbb{E}_{\mathscr{A},\mathbf{1}/n}[\cdot]$ can be factorized in two non-square CPTP isometries

$$\mathbb{E}_{\mathscr{A},\mathbf{1}/n}[\cdot] = \mathcal{J}\mathcal{R}.$$



Given a CPTP map $\mathcal{A}$, its reduction onto the algebra $\mathscr{A}$,

$$\mathcal{A}|_{\mathscr{A}} = \mathcal{R}\mathcal{A}\mathcal{J}$$

is CPTP.

Observable CPTP reduction

We leverage the knowledge of the quantities of interest to reduce the model:

1. Find the **observable subspace** $\mathscr{O}$.
2. Close its orthogonal complement to a ***-algebra** $\mathscr{A} = \mathrm{alg}(\mathscr{O})$;
3. Compute the **cond. exp.** $\mathbb{E}_{\mathscr{A},\mathbf{1}/n}[\cdot]$ and its CPTP factors $\mathcal{R}$ and $\mathcal{J}$;
4. Use the two factors to **reduce the dynamics**:

$$\check{\mathcal{M}}_k = \mathcal{R}\mathcal{M}_k\mathcal{J}, \qquad \check{\mathcal{C}} = \mathcal{C}\mathcal{J}, \qquad \Phi = \mathcal{R}.$$

Note that $\check{\mathcal{M}}_k$ are CP and also $\sum_{k\in\Omega}\check{\mathcal{M}}_k^\dagger(\mathbf{1}) = \mathbf{1}$ as $\mathcal{R}^\dagger, \mathcal{J}^\dagger$ are unital.

Reduction of Measurements and Dynamics

In many cases we have $\mathcal{M}_k = \mathcal{K}_k \circ \mathcal{E}$ with $\mathcal{E}$ a CPTP dynamics and $\mathcal{K}_k(\cdot) = M_k \cdot M_k^\dagger$ a generalized measurement, $\sum_{k\in\Omega} M_k^\dagger M_k = \mathbf{1}$.

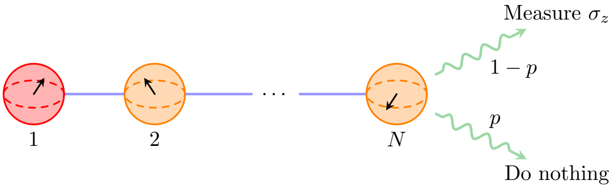
Assuming one of the following:

1. $\exists\{\lambda_k\}$ s.t. $\sum_k \mathcal{M}_k = \mathcal{E}$;
2. $\mathscr{N}^\perp$ is $\mathcal{E}^\dagger$-invariant;
3. $\mathscr{A}$ is $\mathcal{K}_k$-invariant $\forall k$;
4. $\mathscr{A}$ is $\mathcal{E}^\dagger$-invariant;

We have

$$\check{\mathcal{M}}_k = \mathcal{R}\mathcal{M}_k\mathcal{J} = \mathcal{R}\mathcal{K}_k\mathcal{J}\ \mathcal{R}\mathcal{E}\mathcal{J} = \check{\mathcal{K}}_k\check{\mathcal{E}}$$

Monitored spin chain
- A toy model -



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Monitored spin chain - Spaces

For $N \geq 4$, <u>regardeless of N</u> we have $\dim(\mathscr{O}) = 18$.
The evolution of $\rho_1(t)$ is only influences by spins $1, 2, N-1, N$.

Moreover,

$$\mathscr{A} \simeq \mathbb{C}^{4\times 4} \oplus \mathbb{C}^{4\times 4}.$$

and is $\mathcal{K}_k$-invariant.

The reduced model is thus a classical mixture of two two-qubit systems and $\check{\mathcal{M}}_k = \check{\mathcal{K}}_k \check{\mathcal{E}}$.

$$\xi(t) = \left[\begin{array}{c|c} \xi_0(t) & 0 \\ \hline 0 & \xi_1(t) \end{array}\right] \in \mathbb{C}^{8\times 8}$$

Take home ideas

1. We need **algebras** to define a quantum probability space;

2. **Conditional expectations** provide CPTP reduction.

Conclusion

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- **Framework for model reduction of statistical dynamics (CPTP)**
- Applied to discrete-time quantum trajectories **[arXiv:2403.12575]**;
- The framework is very general and has been applied to:
  - (deterministic) Discrete-time case **[arXiv:2307.06319]**;
  - (deterministic) Continuous-time case **NEW!** (joint with LV) **[arXiv:2412.05102]**;
  - Continuous-time quantum trajectories (in preparation).
- **Outlook**
  - Approximate model reduction;
  - Connection with adiabatic elimination techniques;

Thanks for your attention!

Our group's website →